Some specific features of atmospheric tubulence*

By A. M. OBOUKHOV
Institute of Atmospheric Physics, Moscow

(Received 20 October 1961)

The spectrum of atmospheric turbulence is very broad by comparison with spectra in wind tunnels. We introduce the notion of small-scale and large-scale turbulence. Small-scale turbulence consists of a set of disturbances, the scales of which do not exceed the distance to the wall and for which the hypothesis of three-dimensional isotropy is valid in a certain rough approximation. Large-scale turbulence is essentially anisotropic; the horizontal scale in the atmosphere is much larger than the vertical one, the latter being confined to a certain characteristic height H. The horizontal scale varies widely according to the external conditions and characteristics of the medium.

The mean value of the internal scale of turbulence

$$l_1 = (\nu^3/\epsilon)^{\frac{1}{4}} \tag{1}$$

is of the order of 1 cm for the atmosphere. The maximum horizontal scale of turbulent inhomogeneities is of the order of $L_0 = 2500$ km according to different estimates.† The range of the spectrum of atmospheric turbulence in the horizontal direction is

$$L_0/l_1 = 2{\cdot}5 \times 10^8$$

or 28 octaves.

It is well known that the ratio of the external turbulent scale to the internal one is proportional to the Reynolds number to the power 3/4. The fluctuating character of the statistical properties of the small-scale turbulence is due to a very large spectrum width of atmospheric disturbances. As practice shows, it is enough to average the results of the spectral analysis of fluctuations (empirical energy distribution) over the period of 10 min in order to obtain relatively stable spectra of turbulence fluctuations in the surface layer. Using Kolmogorov's theory one can try to approximate these distributions by a relation of the form

$$E(k) = C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}. \tag{2}$$

* *Editors' footnote.* This paper is a record of part of a lecture given by the author at the IUGG–IUTAM Symposium on Fundamental Problems in Turbulence and their Relation to Geophysics, at Marseilles in September 1961. The part reproduced here concerns the local structure of turbulence in general, and was related to a lecture given by A. N Kolmogorov at another symposium in the preceding week (see the following paper in this journal). The two lectures are being included in the published records of the proceedings of the respective Symposia, and are published here also in view of their interest to a large number of English-speaking readers.

† The first estimate of this kind was provided by Defant (1921) on the basis of horizontal mixing study. Modern research on the correlation function behaviour of meteorological elements at distances of the order of some hundred and thousand km give similar values. The external scale in this case is expressed by a rather distinct break in the correlation functions.

As practice shows, one succeeds in the majority of cases with a rather high degree of an accuracy (of the order of 5 %) (Gurvitch 1960). Successive measurements show that, although each measurement is in satisfactory agreement with a $(-5/3)$-power law in a certain range of scales, the intensity of turbulence varies from measurement to measurement, which may be explained by variance of the energy dissipation rate ϵ (the main parameter of the locally isotopic theory). These slow fluctuations of energy dissipation are due to change of the large-scale processes in the observation region, or 'weather' in a general sense. Similar slow macroscopic changes of energy dissipation must be observed at very large Reynolds numbers and they are actually observed in the atmosphere.

The comparatively small statistical stability of the parameter ϵ requires a more distinct determination of a statistical ensemble by which the averaging is made in the theory of locally-isotropic turbulence.*

Let M_1 and M_2 be two observation points located at a distance apart which exceeds the internal scale of turbulence, but is small by comparison with the distance to the boundary of the flow (this is the condition of applicability of a local treatment of turbulence).

Let us consider the rate of dissipation averaged over some volume of a standard form connected invariantly with the points M_1 and M_2. As an example of such a volume V_{M_1, M_2} it is reasonable to take the interior of the sphere with the observation points as poles. Then the average is

$$\tilde{\epsilon}(M_1, M_2) = \frac{1}{\frac{1}{6}\pi r^3} \int_{V_{M_1, M_2}} \epsilon \, dV. \tag{3}$$

Assume that in every measurement of turbulent fluctuations, in particular, in the measurement of an instantaneous difference of velocity $\Delta \mathbf{v} = \mathbf{v}(M_2) - \mathbf{v}(M_1)$, the value $\tilde{\epsilon}$ is recorded simultaneously. Let us now fix the observation points (and the distance r also) and then select only those occasions at which $\tilde{\epsilon}$ takes a certain given value. On repeating this operation many times (under some like external conditions) we obtain the statistical ensemble which could conditionally be called a 'pure' one depending upon the selection of the observation points and the value of $\tilde{\epsilon}$ as parameters.

Now let us compute for this ensemble two-point mean values of momentum introduced by Kolmogorov (1941*a*):

$$B_{nn} = \langle (\Delta \mathbf{v})_n^2 \rangle, \quad B_{dd} = \langle (\Delta \mathbf{v})_d^2 \rangle, \quad B_{ddd} = \langle (\Delta \mathbf{v})_d^3 \rangle.$$

In accordance with the general principle of local similarity these values will depend only upon $\tilde{\epsilon}$, upon the distance r and the internal Reynolds number

$$\tilde{R}(r) = \tilde{\epsilon}^{\frac{1}{3}} r^{\frac{4}{3}} / \nu. \tag{4}$$

For the mean-square of the longitudinal difference of velocities we have the form

$$\langle (\Delta \mathbf{v})_d^2 \rangle_{\tilde{\epsilon}} = C(\tilde{R}) \, \tilde{\epsilon}^{\frac{2}{3}} r^{\frac{2}{3}}. \tag{5}$$

* In the study of turbulence in wind tunnels one usually takes averages over a certain ensemble corresponding to a time ensemble, and the averaging period is larger than the life-time of the largest eddies. In the case of atmospheric turbulence there arise specific difficulties; so that the life-time of the largest eddies in the atmosphere greatly exceeds, as a rule, the time of measuring turbulent characteristics.

The second self-similarity hypothesis of Kolmogorov at rather large Reynolds numbers means in this scheme that if the distances r are large enough, and $\tilde{\epsilon}$ is fixed, the function $C(\tilde{R})$ can be replaced by a constant

$$C_0 = \lim_{\tilde{R}\to\infty} C(\tilde{R}). \tag{6}$$

This assumption will be called a similarity hypothesis for a 'pure' régime.

In practice we always deal with some mixed ensemble in which the value $\tilde{\epsilon}$ varies in accordance with some general statistical law. A natural time ensemble, corresponding to continuous observations of a certain limited part of a turbulent flow for a long enough period of time should be considered as a 'mixed régime'.

Let us consider now a simplified scheme, for which natural means (expectations) of the values required can be computed. Assume that for any choice of the observation points M_1 and M_2 the value $\tilde{\epsilon}$ has a logarithmically normal distribution.* This means that $\tilde{\epsilon}$ can be written in the form

$$\tilde{\epsilon} = \epsilon_0 e^{\eta}, \tag{7}$$

where ϵ_0 is the 'mean geometrical' dissipation value, and η is a random variable obeying the Gaussian distribution law with parameters

$$\overline{\eta} = \langle\eta\rangle = 0, \quad \mathscr{D}(\eta) = \langle\eta^2\rangle = \beta. \tag{8}$$

The logarithmic dispersion β is the main dimensionless parameter characterizing the statistical distribution. For the logarithmic distribution it is not difficult to compute the moment of any order to be

$$\langle\tilde{\epsilon}^{\alpha}\rangle = \epsilon_0^{\alpha} e^{\frac{1}{2}\alpha^2\beta}. \tag{9}$$

In particular, if $\alpha = 1$, we have

$$\bar{\epsilon} = \epsilon_0 e^{\frac{1}{2}\beta}. \tag{10}$$

Note that in contrast to ϵ_0 and β, which depend upon the distance r, the value $\bar{\epsilon}$ does not depend upon the observation distance r; $\bar{\epsilon}$ is the main macroscopic characteristic, which can be obtained, for example, from general considerations of the energy balance of the system.† Further, we shall consider $\bar{\epsilon}$ as a certain given constant.

Now compute mean-square fluctuation of $\tilde{\epsilon}$;

$$\sigma_{\tilde{\epsilon}}^2 = \langle\tilde{\epsilon}^2\rangle - \bar{\epsilon}^2 = \epsilon_0^2(e^{2\beta} - e^{\beta}). \tag{11}$$

The logarithmic dispersion β can be directly expressed in terms of the variance coefficient $M = \sigma/\bar{\epsilon}$;

$$e^{\beta} = 1 + M^2, \tag{12}$$

and

$$\epsilon_0 = \bar{\epsilon}(1 + M^2)^{-\frac{1}{2}}. \tag{13}$$

* This assumption is not very restrictive as an approximate hypothesis since the distribution of any essentially positive characteristic can be represented by a logarithmically Gaussian distribution with correct values of the first two moments (see also Kolmogorov 1941*b*).

† For the study of turbulent structure in a wide range of scales in the atmosphere (flow over an infinite horizontal plane) it is reasonable to average over the horizontal co-ordinates, as the general statistical régime of motion can be considered horizontally homogeneous.

The computation of $\langle \tilde{\epsilon}^{\frac{2}{3}} \rangle$ gives

$$\langle \tilde{\epsilon}^{\frac{2}{3}} \rangle = \epsilon_0^{\frac{2}{3}} e^{2\beta/9}, \tag{14}$$

or, on using (10),

$$\langle \tilde{\epsilon}^{\frac{2}{3}} \rangle = \bar{\epsilon}^{\frac{2}{3}} e^{-\beta/9}. \tag{15}$$

This result can also be written with the aid of (12) in the form

$$\langle \tilde{\epsilon}^{\frac{2}{3}} \rangle = \bar{\epsilon}^{\frac{2}{3}} (1+M^2)^{-\frac{1}{9}}. \tag{16}$$

On averaging Kolmogorov's equation (the '(2/3)-power law'), and taking into account the possible dependence of the variance coefficient M upon the distance r, we obtain for the mixed régime the result

$$\langle (\Delta \mathrm{v})^2_d \rangle = C_0 \bar{\epsilon}^{\frac{2}{3}} \frac{r^{\frac{2}{3}}}{\{1+M^2(r)\}^{\frac{1}{9}}}. \tag{17}$$

To obtain the dependence of M on r some additional assumptions are required.

Proceeding from the fact that the dependence of a longitudinal structure function upon the variance coefficient is very weak (if it exists at all), the dispersion $\mathscr{D}(\tilde{\epsilon})$ can be computed in the first approximation by taking the ordinary (2/3)-power law for the structure function if $r \gg l_1$. The fourth-product moments necessary for computation of the correlation function of disspation can be determined in this case from the Millionshchikov hypothesis (introducing some reduction factor). Thus, the following representation of the correlation function of energy dissipation can be obtained

$$\langle \epsilon'_{M_1} \epsilon'_{M_2} \rangle = \begin{cases} \gamma \bar{\epsilon}^2 & r \ll l_1, \\ \gamma \bar{\epsilon}^2 (l_1/r)^{\frac{8}{3}} & r \gg l_1, \end{cases} \tag{18}$$

where $\epsilon' = \epsilon - \bar{\epsilon}$, and γ and γ' are the numerical coefficients which can depend on the Reynolds number.

The dispersion of averaged dissipation $\mathscr{D}(\tilde{\epsilon})$ can be computed on the basis of this correlation function by known methods. The dependence upon the distance r and other parameters can be described by the following approximate relations:

$$\mathscr{D}(\tilde{\epsilon}) = \begin{cases} \gamma \bar{\epsilon}^2 & r \ll l_1, \\ C\gamma' \bar{\epsilon}^2 (l_1/r)^{\frac{8}{3}} & r \gg l_1, \end{cases} \tag{19}$$

where C is a universal numerical constant.*

By accepting an approximate dependence of the variance coefficient of dissipation upon the distance (if $r \gg l_1$) we obtain the value of the longitudinal structure function in the second approximation with correction for dissipation variance.

On expressing the internal scale in terms of the external scale and the Reynolds number, the result obtained can be represented in the form

$$\langle (\Delta \mathrm{v}^2)_d \rangle = \frac{C_0 \bar{\epsilon}^{\frac{2}{3}} r^{\frac{2}{3}}}{\left\{1 + \frac{C\gamma'}{R^2} \left(\frac{L_0}{r}\right)^{\frac{8}{3}}\right\}^{\frac{1}{9}}}. \tag{20}$$

* The question of dispersion of dissipation is treated in more detail in the work of Golitsyn (1962).

The computation given above is, of course, of a tentative character. However, it shows that for the mixed régime (natural time ensemble) one can notice a very weak deviation from the (2/3)-power law if Kolmogorov's hypotheses are considered valid for the 'pure' ensemble describing the local régime of turbulence at a given dissipation value. Another alternative is that the 'mixed' régime is asymptotically described by the (2/3)-power law, in which case the local régime with definite dissipation cannot strictly satisfy the second Kolmogorov hypothesis (self-similarity with respect to internal Reynolds number).

REFERENCES

DEFANT, A. 1921 Die Bestimmungen der Turbulenz grössen der atmosphärischen Zirkulation aussertropischer Breiten. *S.B. Akad. Wiss. Wien* **130**, 383.

GOLITSYN, G. S. 1962 Fluctuations of dissipation in the local isotropy flow. *Dokl. Akad. Nauk. SSSR.* (to be published).

GURVITCH, A. S. 1960 Experimental research on frequency spectra of atmospheric turbulence. *Izv. Akad. Nauk. SSSR, geofiz. ser.*, no. 7, 1042.

KOLMOGOROV, A. N. 1941*a* Local structure on turbulence in incompressible fluid. *C.R. Acad. Sci. U.R.S.S.* **30**, 299.

KOLMOGOROV, A. N. 1941*b* On logarithmically-normal law of distribution. *C.R. Acad. Sci. U.R.S.S.* **31**, 99.